

Outer space and Automorphisms of free groups

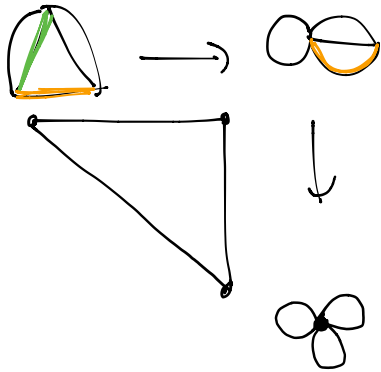
LECTURE 6

We're studying $K_n =$ spine of outer space
= contractible, cocompact simplicial complex

Today: Cube complex structure, homology computations and connection with Kontsevich's graph homology.

In graphical description of K_n , simplex
= chain of forest collapses

$$\begin{array}{ccccccc} G & \longrightarrow & G_{\varphi_1} & \longrightarrow & (G_{\varphi_1})_{\varphi_2} & \longrightarrow & \dots \longrightarrow & (\dots (G_{\varphi_1})_{\varphi_2} \dots)_{\varphi_k} \\ \uparrow & & \nearrow & & \nearrow & & \nearrow & \\ R_n & & & & & & & \end{array}$$

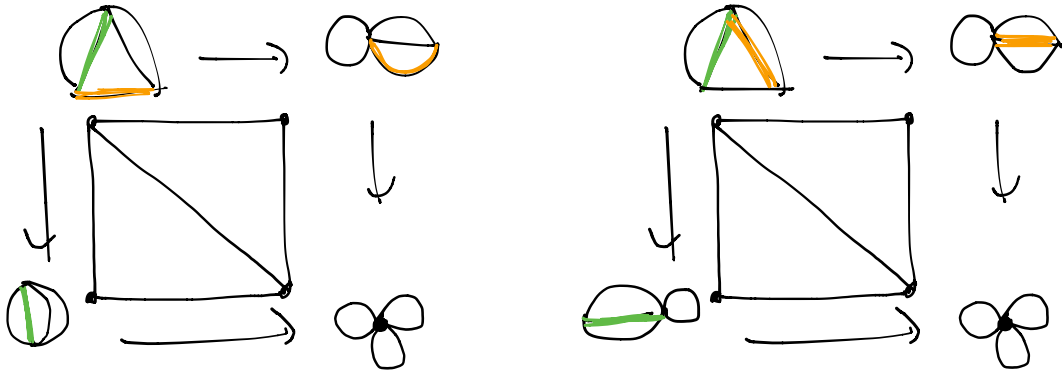


Can regard this as a chain of forests in the first graph

$$(G, \Phi_1 \subset \Phi_2, g)$$

where $\Phi_i = \text{inverse image of } \varphi_i$

Given $\Phi \subset G$, can collapse the edges one at a time in any order to get a cube in K_n :



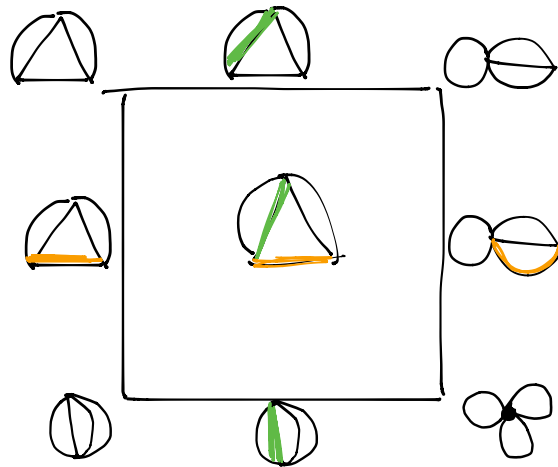
This cube can be oriented by ordering the edges of Φ .

K_n is a cube complex with cubes $c(g, G, \Phi)$,


Φ a forest in G

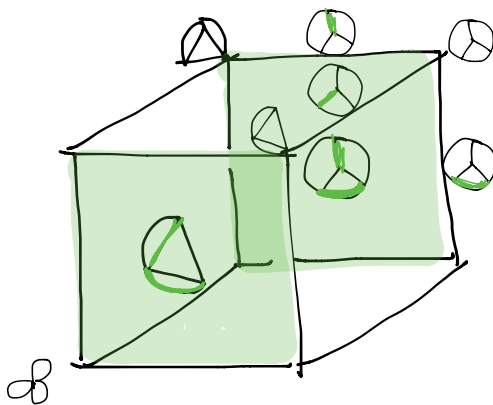
$$\underline{\dim} c(g, G, \Phi) = \# \text{ edges of } \Phi.$$

Faces of $c(g, G, \Phi)$:



$$\partial(G, \Phi, g) = \sum_{e \in \Phi} (G, \Phi - e) - (G_e, \Phi_e)$$

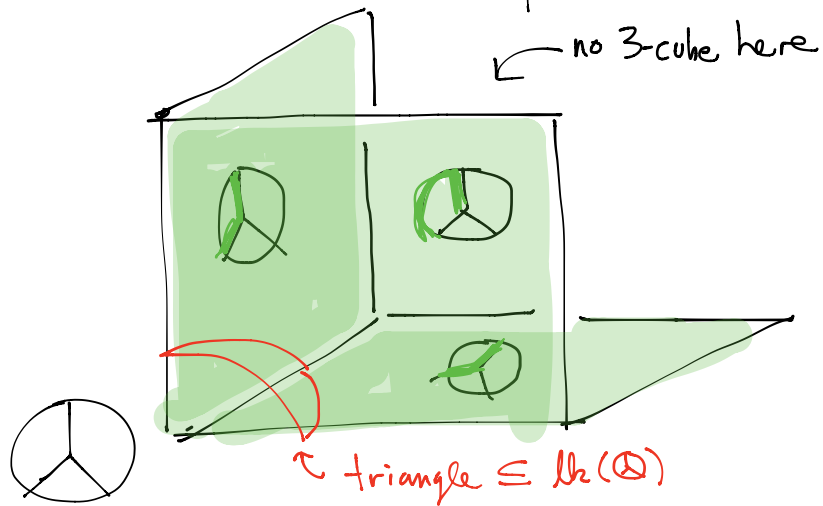
 = 3-cube:



Aside:

Recall a cube complex is NPC (locally CAT(0))
if links of vertices are flag complexes.

K_4 is not a CAT(0) cube complex:



Exercise: $\text{Lk}(v)$ is homotopic to a
wedge of 2-spheres. How many?

Can't even fudge the angles to make a CAT(0)
cube complex (Bridson's thesis), and in fact

Thm (Bridson) $\text{Out}(F_n)$ cannot act properly and
cocompactly on a CAT(0) metric space.

Recall: $\Gamma < \text{Out}(F_n)$ torsion-free, finite index

$$\Rightarrow H^*(\Gamma) = H^*(K_n/\Gamma)$$

This is with any coefficients

If we take trivial coeffs in \mathbb{Q} (or any field of char 0), turns out

$$H^*(K_n/\text{Out}F_n; \mathbb{Q}) \cong H^*(\text{Out}F_n; \mathbb{Q})$$

Heuristically: $H^*(H; \mathbb{Q}) = 0$ if H is finite,
so $H^*(\cdot; \mathbb{Q})$ doesn't "see" the finite stabilizers,
thinks the action is free.

eg $H = \mathbb{Z}/n\mathbb{Z}$ has chain complex

$$\begin{array}{ccccccccccc} \dots & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \rightarrow & 0 \\ & & & & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & & & & \\ \text{tensor by } \mathbb{Q}: & & & & & & & & & & & & & & & & & \\ \dots & \rightarrow & \mathbb{Q} & \xrightarrow{\times n} & \mathbb{Q} & \xrightarrow{0} & \mathbb{Q} & \xrightarrow{\times n} & \mathbb{Q} & \xrightarrow{0} & \mathbb{Q} & \xrightarrow{\times n} & \mathbb{Q} & \xrightarrow{0} & \mathbb{Q} & \rightarrow & 0 \end{array}$$

$\cong \qquad \cong \qquad \cong \qquad \cong$

Formally: There is a spectral sequence corresponding to the action of $\text{Out} F_n$ on K_n , converging to $H^*(\text{Out} F_n)$; E^2 -terms correspond to vertex stabilizers if $q > 0$, so vanish with trivial \mathbb{Q} -coefficients, spectral sequence collapses to a chain complex for $H^*(K_n/\text{Out} F_n)$

(Details: K. Brown's book, equivariant homology spectral sequence, (Chap. VII, Sec. 7))

So we can compute $H^*(\text{Out} F_n; \mathbb{Q})$ by computing $H^*(K_n/\text{Out} F_n; \mathbb{Q})$.

Action of $\text{Out}(F_n)$ on K_n changes the marking $(G, g) \cdot \psi = (G, g \circ f)$, in fact acts transitively on markings of G .

So $K_n / \text{Out}(F_n)$ has one vertex for each isomorphism class of graphs G

And one k -simplex for each isomorphism class of chains of k forests $(G, \Phi_1 \subset \dots \subset \Phi_k)$

e.g. If T, T' are (non-equivalent) maximal

trees in G , then $G \longrightarrow G_T \approx R$

and $G \longrightarrow G_{T'} \approx R$

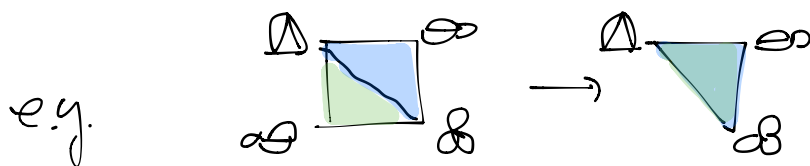
are two different edges from G to the root R

So $K_n / \text{Out}(F_n)$ is a union of simplices, but not a simplicial complex.

The ∂ is a mess to compute. Easier if you use the cube complex structure:

BUT: $K_n / \text{Out}(F_n)$ is not a union of cubes:

An auto of (G, Φ) induces an auto of F_n which folds the cube:



The stabilizer of the cube $c(G, \Phi, g)$ is isomorphic to $\text{Aut}(G, \Phi)$ (graph isomorphisms preserving Φ)

Prop The quotient $c(G, \Phi, g) / \text{stab}(G, \Phi, g)$ is a cone on a rational homology sphere or disk, depending on whether $\text{Aut}(G, \Phi)$ contains an orientation-reversing element.

PF: Look at equivariant homology spectral

sequence: G acts on X . Two filtrations of

$C_* X \otimes_{\mathbb{Z}G} C_* EG$ give two spectral

sequences converging to something

$$E_{p,q}^2 = H_p(G; H_q X)$$

and $E_{p,q}^2 = H_p(X/G, H_q) = \bigoplus_{G\text{-orbits}} H_q(G_{\sigma_i})$

For G finite, rat'l coeffs, $X = S^k$, get

k	$H_0(G; H_k(S^k))$	0	0	0	<div style="background-color: yellow; padding: 5px; display: inline-block;">This is $E_{p,q}^1$!</div>
	0	0	0	0	
	0	0	0	0	
	$H_0(G)$	0	0	0	
					$\bigoplus_{\sigma_0} \mathbb{Q} \leftarrow \bigoplus_{\sigma_1} \mathbb{Q} \leftarrow \bigoplus_{\sigma_2} \mathbb{Q} \leftarrow \dots$

\uparrow
 $= 0$ except in $\dim 0, k$.

$= 0$ in $\dim k$ if G reverses orientation.

\uparrow
 computes $H_*(S^k/G)$

Cor $H_*(K_n / \text{Out } F_n)$ is the homology of the chain complex C_* with $C_k = \bigoplus_{(G, \Phi)} \mathbb{Q}$

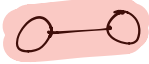

where • G has rank 2

• Φ is a forest in G with 2 edges

• (G, Φ) has no orientation-reversing

automorphisms

$$\text{and} \cdot \partial(G, \Phi) = \sum_{e \in \Phi} (G, \Phi - e) - (G_e, \Phi_e) \\ = \partial_p(G, \Phi) - \partial_c(G, \Phi)$$

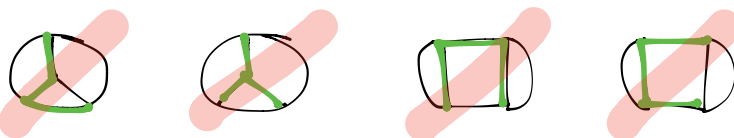
eg: $n=2$. vertices $\textcircled{1} \quad \infty$ 
edges 
↑ remove separating edges

(no or-reversing autos!) quotient 

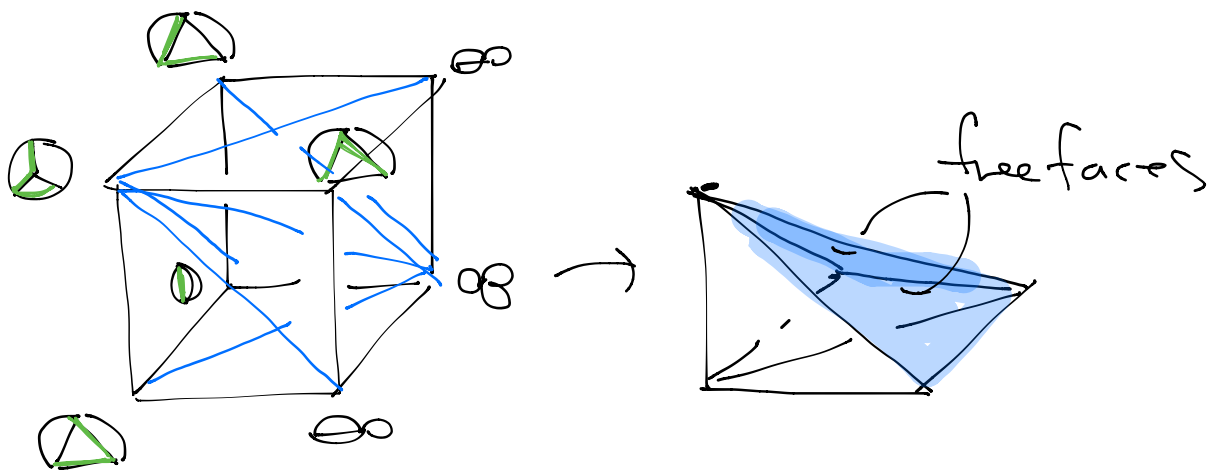
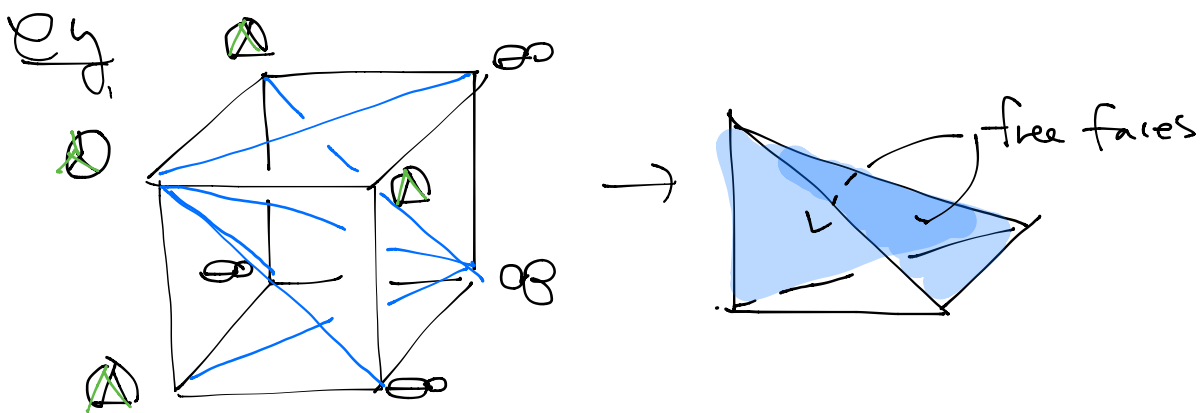
$$\partial(\textcircled{1}) = \textcircled{1} - \infty$$

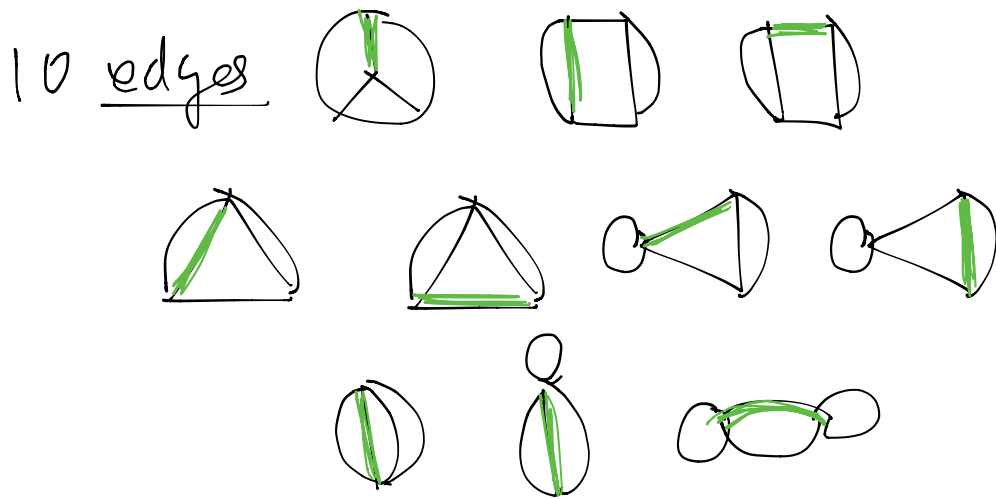
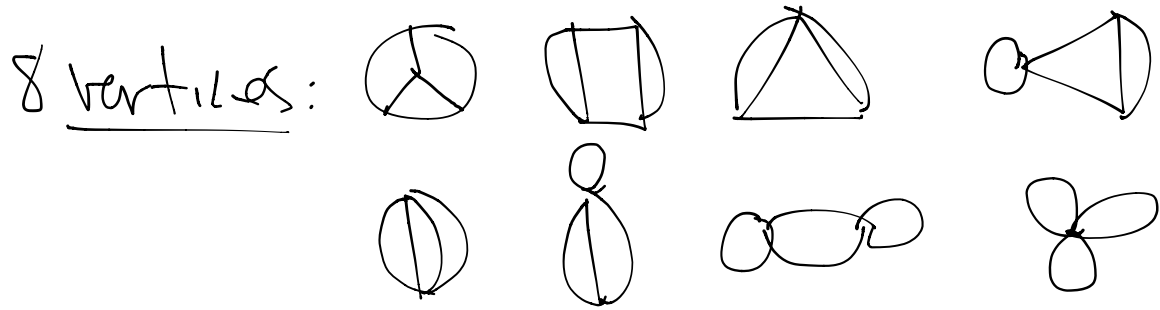
So $H^*(\text{Out } F_2; \mathbb{Q}) = 0$ for $* > 0$.

$H^*(\text{Aut}F_3)$: K_4 is 3 dimensional

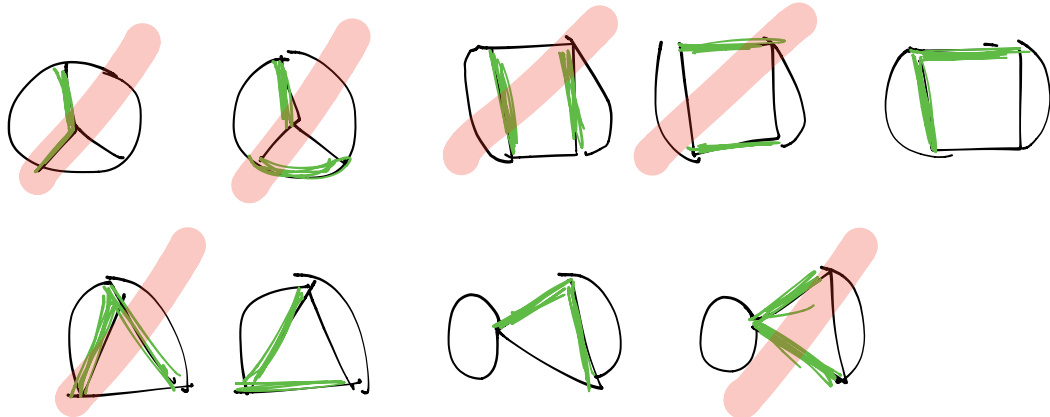
3 cubes: 

Each has an orientation-reversing automorphism, so
a free face in $K_3/\text{Aut}F_3$, so $H^3 = 0$





2-cubes



Chain complex

$$C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

$$0 \rightarrow \mathbb{Q}^3 \rightarrow \mathbb{Q}^{10} \rightarrow \mathbb{Q}^8 \rightarrow 0$$

Exercise: Compute ∂ maps, show

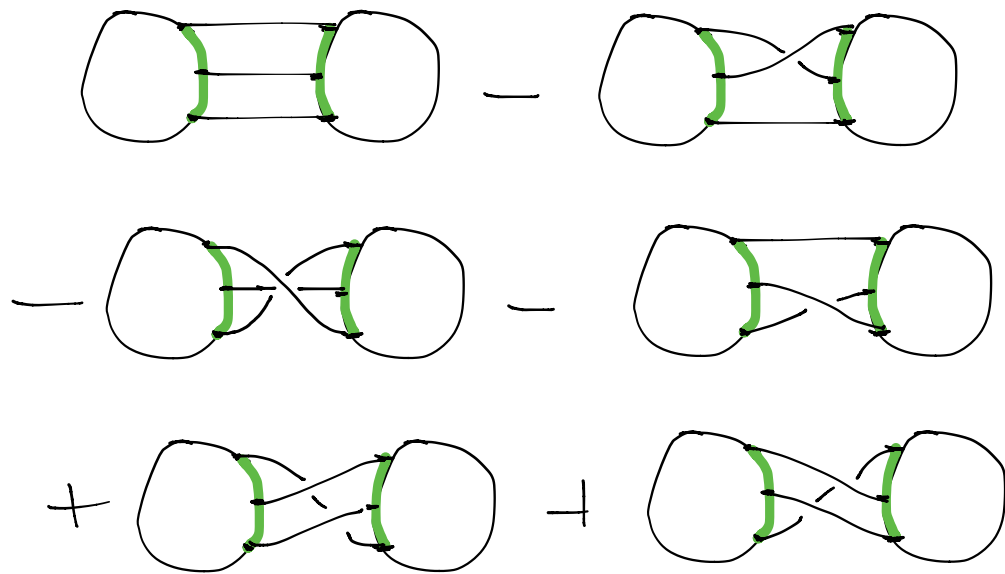
$$H_x = 0, \quad x > 0$$

$n=4$ is about the limit of reasonable
computability

$$\text{Get } H_1 = H_2 = H_3 = H_5 = 0 \quad \underline{\underline{\text{but}}}$$

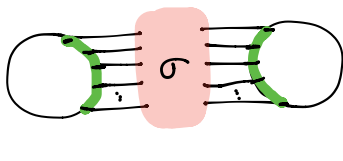
$$H_4(\text{Out } F_4) \cong \mathbb{Q}!$$

Here is the 4-cycle which generates $H_4(\text{Out } F_4)$:
 it is made out of (quotients of) 4-dimensional
 cubes



$$= \sum_{\sigma \in \Sigma_3} \text{sign}(\sigma) \cdot \text{Diagram}(\sigma)$$

Easy to generalize:

$$\mu_k = \sum_{\sigma \in \Sigma_{2k+1}} \text{sign}(\sigma) \cdot \text{Diagram}$$


This is a cycle, called the k -th Morita cycle

μ_1, μ_2, μ_3 are not boundaries

Conjecture (Morita) μ_k represents a non-zero homology class for all k .

Morita found these cycles via the relation with Kontsevich's graph homology of the Lie operad

To make the connection, need to simplify our chain complex.

We have

$$C_k = \bigoplus_{\substack{(G, \Phi) \\ e(\Phi) = k}} \mathbb{Q} = \bigoplus_{e(\Phi) = k} (G, \Phi)$$

We can decompose this further by # vertices of G

$$C_k = \bigoplus_{p+q=k} C_{p,q}$$

$$\text{where } C_{p,q} = \bigoplus_{\substack{vG=p \\ e\Phi=p+q}} (G, \Phi)$$

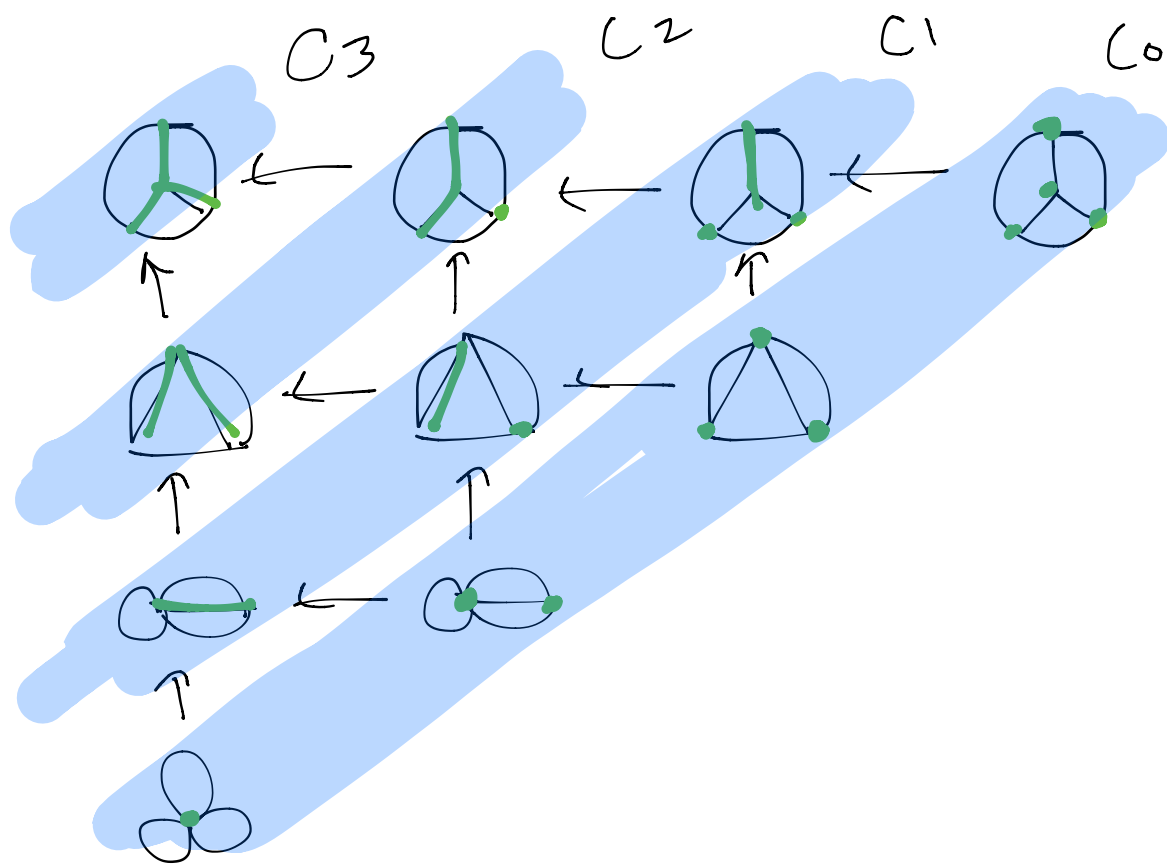
Top-dimil cubes are

$$(G, T) = (\text{trivalent graph, maximal tree}) \\ (vG = 2n-2, e\Phi = 2n-3)$$

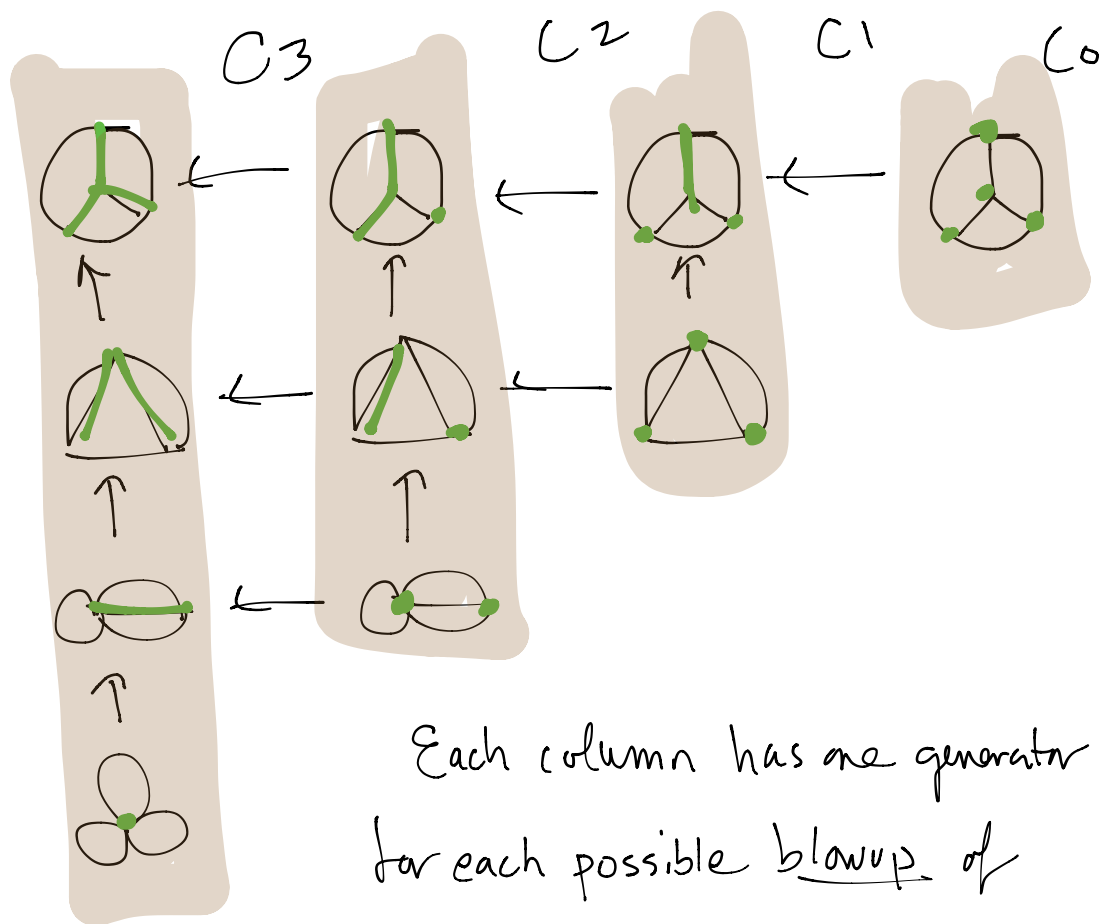
$$\text{next are } vG = 2n-2, e\Phi = 2n-4$$

$$\text{and } vG = 2n-3, e\Phi = 2n-4$$

etc.



We can compute $H^*(C_*)$ by first computing the vertical homology, then the horizontal H_* (this is spectral sequences, but don't tell anybody)



Each column has one generator for each possible blowup of the graphs G at the bottom

This is the chain complex for $lk_{>G}$.

We know $lk_{>G} \simeq \bigvee S^{v(G)-2}$, so has no homology except in the top dimension

So same is true for $lk_{>G}/Aut(G) \subset K_n/Aut(F_n)$

(again by equivariant H_* spectral sequence)